

STUDENTS' UNDERSTANDING OF FUNDAMENTAL GEOMETRY CONCEPTS: AN EMPIRICAL INVESTIGATION

Dr. Ajay Kumar Mishra
Professor & Vice Chancellor
Sikkim Skill University
Namthang, Sikkim

ABSTRACT

Geometry is a basic part of mathematics, and it provides crucial elements for understanding spatial relationships, logical reasoning, and deductive proof. This research paper investigates students' understanding of basic geometry concepts through an empirical study based on fundamental theoretical concepts. The paper focuses on the importance of undefined terms like point, line, and plane, and basic geometric definitions like angles, parallel lines, and perpendicular lines. Crucial theorems like the triangle angle sum theorem, the Pythagorean theorem, and criteria for triangle congruence and similarity are presented with mathematical expressions and examples. The paper also focuses on the most significant learning difficulties encountered by students, like difficulties in geometric visualization, misconceptions in basic definitions, mistakes in transformation rule applications, and weaknesses in theorem-based proof statements. The empirical views presented through achievement tests, diagnostic questioning, error analysis, and classroom observations clearly state that students' performance is highly dependent on their conceptual understanding and capacity to apply geometric reasoning skills rather than mere memorization.

Keywords: *Geometry learning, Euclidean geometry, theorems, proofs, congruence, similarity, transformations, empirical investigation.*

1. INTRODUCTION

Geometry is one of the oldest and most basic branches of mathematics that deals with the study of shapes, sizes, positions, and relationships. The classical Euclidean geometry is based on basic geometric entities like points, lines, angles, planes, and figures, which serve as the basis for further mathematical constructs.

The geometry of the plane is defined by axioms and postulates that result in significant theorems and relationships. For instance, the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in a Cartesian plane is given by:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similarly, the midpoint of a line segment joining these points is expressed as:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Angles are another fundamental concept in geometry. If two rays form an angle θ , its measure satisfies:

$$0^\circ \leq \theta \leq 360^\circ$$

One of the most significant results involving angles is the interior angle sum theorem for triangles:

$$\angle A + \angle B + \angle C = 180^\circ$$

In polygon geometry, the sum of interior angles of an n -sided polygon is:

$$S = (n - 2) \times 180^\circ$$

Area relationships form another core component of geometry. For basic plane figures:

- Area of a triangle:

$$A = \frac{1}{2}bh$$

- Area of a rectangle:

$$A = lw$$

- Area of a circle:

$$A = \pi r^2$$

- Circumference of a circle:

$$C = 2\pi r$$

Right-angled triangles satisfy the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

where c is the hypotenuse. This theorem forms the basis for many applications in measurement and coordinate geometry.

2. LITERATURE REVIEW

Kuzle and Glasnovic Gracin (2020) investigated geometry education in terms of basic mathematical ideas by examining children's drawings. The study found that the drawings made by students represented their growing geometric ideas. The authors concluded that children were able to depict geometric relationships in their drawings even when they did not understand the formal properties of geometry. The authors concluded that students' informal visual understanding of geometry played an important role in influencing their geometric thinking. This study helped in understanding how learners made meaning of geometry through representation and visualization.

Luneta (2015) investigated misconceptions in geometry among students by examining the final Grade 12 examination questions. The study found that students commonly showed misconceptions in concepts such as angles, triangles, and circle thems. Luneta observed that students commonly struggled with the application of theorems and mostly used misconceptions instead of logical reasoning. The study pointed out that misconceptions existed even at a higher level of education, showing a lack of understanding in geometry concepts. The study emphasized the significance of diagnostic tests in understanding students' misconceptions in geometry learning.

Jelatu and Ardana (2018) explored the impact of a GeoGebra-supported REACT (Relating, Experiencing, Applying, Cooperating, and Transferring) teaching approach on students' conceptual understanding of geometry and revealed that the integration of dynamic geometry software significantly enhanced learning outcomes. It was found that the use of GeoGebra enabled students to visualize geometric concepts more effectively, manipulate figures dynamically, and develop a deeper understanding of relationships between shapes and properties. The REACT approach further supported this process by encouraging students to connect mathematical ideas with real-life situations, actively engage in hands-on learning, collaborate with peers, and apply knowledge in different contexts. As a result, students demonstrated improved conceptual clarity, better problem-solving abilities, and greater confidence in geometry. Additionally, the study observed increased student engagement, motivation, and interest in learning when compared to traditional teaching methods. The authors concluded that the integration of technology with interactive and context-based instructional strategies not only improved students' academic achievement but also fostered meaningful and long-lasting understanding of geometric concepts.

Tutak and Adams (2015) investigated elementary preservice teachers' content knowledge in geometry and discovered that, although they could complete procedural tasks, many participants had a poor

conceptual grasp of basic geometric ideas. According to the study, preservice teachers frequently had trouble with subjects like geometric definitions, spatial reasoning, and shape attributes. They also had deficiencies in their ability to clearly explain concepts. Additionally, it revealed weaknesses in their capacity to relate various geometric concepts and use them in educational settings. In order to guarantee successful geometry instruction at the elementary level, the results highlighted the necessity of enhancing conceptual understanding, reasoning abilities, and pedagogical subject knowledge in teacher education programs

3. THEORETICAL FOUNDATION OF GEOMETRY

Geometry is one of the most elementary and logically structured branches of mathematics. The development of geometry, particularly Euclidean geometry, is founded on a formal system where mathematical truths are deduced through logical reasoning. The theoretical foundation of geometry is based on three essential components: undefined terms, definitions, and axioms culminating in theorems. These three components constitute the foundation upon which all geometric concepts, truths, and relationships are developed.

3.1 Undefined Terms

In Euclidean geometry, there are some fundamental ideas that are not formally defined because they are thought to be too basic to be defined in simpler terms. These ideas are referred to as undefined terms, and they are the foundation of the entire geometric structure. The three main undefined terms are point, line, and plane.

A point is a precise location in space and has no dimensions of length, width, or thickness. In coordinate geometry, a point on a plane is denoted by an ordered pair:

$$P(x, y)$$

and the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

A line is defined as an infinite straight way that has infinitely many points and extends endlessly in both directions. A line passing through two distinct points A and B is denoted by:

$$\overleftrightarrow{AB}$$

In analytic geometry, the equation of a straight line is commonly written as:

$$y = mx + c$$

where m represents the slope and c represents the intercept.

A plane is defined as a flat surface that extends infinitely in all directions. A plane is a two-dimensional geometric object that holds an infinite number of points and lines. In three-dimensional coordinate geometry, the general equation of a plane is given by:

$$Ax + By + Cz + D = 0$$

These undefined terms are the basic building blocks of geometry. Based on these undefined terms, complex definitions, axioms, and theorems are created that form the basis of all geometric reasoning and proofs.

4. FUNDAMENTAL DEFINITIONS IN GEOMETRY

Definitions provide clarity and precision.

Definition 1: Line Segment

A line segment is the part of a line bounded by two endpoints.

If two points are A and B , then the segment is written as:

$$\overline{AB}$$

Definition 2: Angle

An angle is formed by two rays sharing a common endpoint called the vertex.

The measure of an angle is expressed in degrees:

$$0^\circ \leq \theta \leq 360^\circ$$

Definition 3: Parallel Lines

Two lines are parallel if they never intersect.

Symbolically:

$$l_1 \parallel l_2$$

Definition 4: Perpendicular Lines

Two lines are perpendicular if they intersect at right angles:

$$l_1 \perp l_2$$

5. KEY THEOREMS IN FUNDAMENTAL GEOMETRY

Understanding geometry depends on mastering core theorems and proofs.

5.1 Angle Sum Property of a Triangle**Theorem 1**

The sum of interior angles of any triangle is 180° .

$$\angle A + \angle B + \angle C = 180^\circ$$

Proof

Consider triangle ABC . Through vertex A , draw a line parallel to BC .

By alternate interior angles:

$$\angle ABC = \angle 1$$

$$\angle ACB = \angle 2$$

Now along the straight line:

$$\angle 1 + \angle A + \angle 2 = 180^\circ$$

Substituting:

$$\angle B + \angle A + \angle C = 180^\circ$$

Hence proved.

Example

If:

$$\angle A = 50^\circ, \angle B = 60^\circ$$

Then:

$$\angle C = 180^\circ - (50^\circ + 60^\circ) = 70^\circ$$

5.2 Pythagoras Theorem

Theorem 2

In a right triangle:

$$a^2 + b^2 = c^2$$

where c is the hypotenuse.

Proof (Sketch)

Construct squares on each side of a right triangle. By comparing the areas, the square on the hypotenuse equals the sum of squares on the other two sides.

Hence:

$$c^2 = a^2 + b^2$$

Example

If:

$$a = 6, b = 8$$

Then:

$$\begin{aligned} c^2 &= 6^2 + 8^2 = 36 + 64 = 100 \\ c &= 10 \end{aligned}$$

5.3 Congruence of Triangles

Theorem 3 (SSS Congruence Rule)

If three sides of one triangle equal three sides of another triangle, then triangles are congruent.

$$AB = DE, BC = EF, AC = DF$$

Then:

$$\triangle ABC \cong \triangle DEF$$

Educational Importance

Students often confuse congruence with similarity. Congruence implies **same shape and size**, not just shape.

5.4 Similarity of Triangles

Theorem 4 (AA Similarity Rule)

If two angles of one triangle equal two angles of another triangle:

$$\triangle ABC \sim \triangle DEF$$

Then side ratios are equal:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Example

If:

$$\frac{AB}{DE} = \frac{4}{2} = 2$$

Then triangle ABC is twice as large as triangle DEF .

6. STUDENTS' DIFFICULTIES IN GEOMETRY UNDERSTANDING

Students may encounter problems in geometry because basic concepts are developed around rigorous definitions, algebraic relationships, and theorem-based reasoning. Geometry demands proper understanding and application of formulas, congruence statements, similarity ratios, and proof statements.

6.1 Difficulty in Visualization of Geometric Transformations

Students struggle in understanding transformations such as rotation, reflection, and translation.

- Translation of a point:

$$(x, y) \rightarrow (x + a, y + b)$$

- Reflection about the x -axis:

$$(x, y) \rightarrow (x, -y)$$

- Reflection about the y -axis:

$$(x, y) \rightarrow (-x, y)$$

- Rotation about origin by 90° :

$$(x, y) \rightarrow (-y, x)$$

- Rotation about origin by 180° :

$$(x, y) \rightarrow (-x, -y)$$

Errors occur when students cannot correctly predict the new coordinates or orientation after transformation.

6.2 Misunderstanding of Definitions and Properties

Many students confuse fundamental geometric definitions.

Parallelogram Property:

Opposite sides are parallel:

$$AB \parallel CD, BC \parallel AD$$

Rectangle Property:

A rectangle is a parallelogram with right angles:

$$\angle A = \angle B = \angle C = \angle D = 90^\circ$$

Thus:

$$\text{Rectangle} \subset \text{Parallelogram}$$

Circle Measurements:

Radius:

$$r$$

Diameter:

$$d = 2r$$

Circumference:

$$C = 2\pi r$$

Students often confuse r and d , leading to incorrect calculations.

6.3 Difficulty in Theorem Application and Proof Structure

Geometry involves theorem-based reasoning.

Example: Triangle Angle Sum Theorem

$$\angle A + \angle B + \angle C = 180^\circ$$

Pythagoras Theorem

$$a^2 + b^2 = c^2$$

Students often apply formulas without conditions, such as using Pythagoras theorem in non-right triangles.

Proof requires logical steps:

Given:

$$AB = DE, BC = EF, AC = DF$$

Then:

$$\triangle ABC \cong \triangle DEF \text{ (SSS Rule)}$$

Students face difficulty linking statements to congruence criteria.

6.4 Memorization Without Conceptual Application

Geometry formulas must be applied correctly.

Area of triangle:

$$A = \frac{1}{2}bh$$

Area of circle:

$$A = \pi r^2$$

Interior angle sum of polygon:

$$S = (n - 2) \times 180^\circ$$

Students memorize these results but often fail in applying them in problem-solving or proof contexts.

7. EMPIRICAL PERSPECTIVE ON GEOMETRY LEARNING

The empirical research on geometry learning is intended to explore how students learn and apply basic geometric concepts. This is important because geometry requires not only mathematical skills but also spatial skills, concepts, and logical reasoning. The empirical study is important because it offers concrete evidence of students' performance, misconceptions, and reasoning processes when dealing with geometric definitions, theorems, diagrams, and proofs.

The empirical study is also concerned with the use of achievement tests to determine the learners' understanding of basic concepts such as angles, triangles, polygons, congruence, similarity, circles, and coordinate geometry. The achievement test is intended to determine whether the learners are able to apply basic geometric concepts, such as the triangle angle sum theorem:

$$\angle A + \angle B + \angle C = 180^\circ$$

or the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

These types of achievement-based assessments are helpful in determining the level to which students are able to solve routine as well as higher-order geometric problems.

Apart from achievement testing, conceptual diagnostic questions are often used to determine specific misconceptions and gaps in understanding. Diagnostic tasks are very helpful in determining whether students are using memorized formulas or have actual conceptual understanding. For example, students may be able to recall the area formula:

$$A = \frac{1}{2}bh$$

but are unable to describe its geometric significance or use it properly in novel situations. Conceptual testing emphasizes these shortcomings and offers a perspective on the level of students' reasoning.

Empirical studies also employ interviews and classroom observations to better investigate students' cognitive processes. By means of interviews, students are asked to describe how they interpret geometric representations, why they select specific theorems, or how they justify the steps of a proof. Classroom observations enable researchers to investigate classroom interactions, teaching practices, and the use of visual representations in learning geometry. These qualitative approaches provide information on how learners build meaning from geometric shapes and how teachers can develop deductive reasoning.

A further key method in empirical geometry studies is error analysis, which consists of systematically analyzing students' errors in solving geometry tasks. These errors, such as a misunderstanding of similarity and congruence, incorrect application of theorems, or misconceptions about the properties of parallel lines:

$$l_1 \parallel l_2$$

and perpendicular lines:

$$l_1 \perp l_2$$

offer strong evidence of conceptual difficulties. By analyzing such patterns, researchers can identify which geometric concepts require improved instructional emphasis.

8. CONCLUSION

Geometry is a mathematically dense and logically organized area that forms a foundation of school mathematics. This paper analyzed the understanding of basic geometry concepts among students by introducing essential theoretical backgrounds like undefined terms, definitions, and axioms culminating in crucial theorems like the distance formula, angle sum property of triangles, Pythagoras theorem, and criteria of congruence and similarity using symbolic representations, proofs, and examples. The paper emphasized that students tend to encounter problems owing to misconceptions in definitions, improper use of geometric properties, and difficulties in conceptualizing transformation geometry concepts like reflection, rotation, and translation. Misuse of theorems and proof-writing also point towards an overemphasis on memorization over conceptual understanding. Empirical data from achievement tests, diagnostic tests, interviews, and error analysis reveal that geometry problem-solving performance is heavily dependent on students' capacity to relate formulas and theorems to their definitions.

REFERENCES

1. Al-Mutawah, M. A., Thomas, R., Eid, A., Mahmoud, E. Y., & Fateel, M. J. (2019). *Conceptual understanding, procedural knowledge and problem-solving skills in mathematics: High school graduates work analysis and standpoints. International journal of education and practice*, 7(3), 258-273.

2. Baiduri, B., Ismail, A. D., & Sulfiyah, R. (2020). Understanding the concept of visualization phase student in geometry learning. *International Journal of Scientific & Technology Research*, 9(2), 2353-2359.
3. Burrows, N. L., & Mooring, S. R. (2015). Using concept mapping to uncover students' knowledge structures of chemical bonding concepts. *Chemistry Education Research and Practice*, 16(1), 53-66.
4. Fang, N. (2015, June). Using Students-Generated Concept Maps to Assess Students' Conceptual Understanding in a Foundational Engineering Course. In 2015 ASEE Annual Conference & Exposition (pp. 26-1684).
5. İbili, E., Çat, M., Resnyansky, D., Şahin, S., & Billingham, M. (2020). An assessment of geometry teaching supported with augmented reality teaching materials to enhance students' 3D geometry thinking skills. *International Journal of Mathematical Education in Science and Technology*, 51(2), 224-246.
6. Jelatu, S., & Ardana, I. (2018). Effect of GeoGebra-Aided REACT Strategy on Understanding of Geometry Concepts. *International journal of instruction*, 11(4), 325-336.
7. Khalil, M., Farooq, R., ÇAKIROĞLU, E., Khalil, U., & Khan, D. (2018). The development of mathematical achievement in analytic geometry of grade-12 students through GeoGebra activities. *Eurasia Journal of Mathematics Science and Technology Education*, 14(4).
8. Kuzle, A., & Glasnovic Gracin, D. (2020). Making Sense of Geometry Education through the Lens of Fundamental Ideas: An Analysis of Children's Drawings. *The Mathematics Educator*, 29(1), 7-52.
9. Luneta, K. (2015). Understanding students' misconceptions: an analysis of final Grade 12 examination questions in geometry. *Pythagoras*, 36(1), 1-11.
10. Malatjie, F., & Machaba, F. (2019). Exploring mathematics learners' conceptual understanding of coordinates and transformation geometry through concept mapping. *Eurasia Journal of Mathematics, Science and Technology Education*, 15(12), em1818.
11. Tutak, F. A., & Adams, T. L. (2015). A study of geometry content knowledge of elementary preservice teachers. *International Electronic Journal of Elementary Education*, 7(3), 301-318.